Engineering Tripos Part IA **First Year**

Paper 2 - MATERIALS

HANDOUT 4

Section 6 covers Examples Paper 3, Q.9-12
Section 7 covers Examples Paper 4, Q.1-7

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Background Web resources:
Section 6: www.doitpoms.ac.uk/tlp/dislocations
Section 7: www.aluminium.matter.org.uk
(see: *Materials Science and Engineering*:
Mechanical Properties – Introduction to Property Charts
Case Studies – Bicycle Design)

6. Microstructure of Engineering Materials II

6.1 Atomic basis of Plasticity in Crystalline Materials

Recall that in *crystalline materials*, the key features of *atomic packing* are:
- atoms/ions pack together as *hard spheres*
- they pack in planes, which stack to form the *lattice*
- lattices are *close-packed* (FCC, HCP), or nearly so (BCC)
- straight lines of touching atoms form *close-packed directions*.

The *atomic bonding* is strong and primary: *metallic, ionic or covalent*.

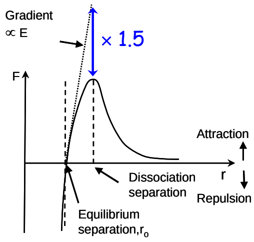
Elastic deformation displaces atoms by a fraction of their equilibrium spacing.

Plastic deformation involves relative movement of material over very large multiples of the atomic spacing.

The issues therefore are:
- how is this achieved at the atomic level?
- can the behaviour be manipulated to increase material strength?

6.1.1 Ideal Strength of Crystalline Material

Estimate of upper limit on strength from atomic force-distance curve:



- bonds rupture at the *dissociation separation*, of order 1.1 × r_o, i.e. a *strain* of approx. 10%.
- force – extension curve is *linear near equilibrium separation*, and is shown extrapolated in the figure.
- notional linear-elastic force at separation of 1.1 r_o is higher than the peak force – by a factor of order 1.5.

Tensile stress needed to break all the bonds simultaneously is thus of order 1/1.5 of a notional elastic stress at a strain of 10%:

$$\sigma_{peak} \approx 1/1.5 \times E \times 0.1 \approx E/15$$

This is an estimate of the *ideal strength* of a material.

How does this order of magnitude estimate for *ideal strength* compare with the *actual strength*?

Typical data for $\left(\dfrac{\text{elastic limit}}{\text{Young's modulus}}\right)$:
- Glass, diamond ≈ 1/40 – 1/20
- Ceramics ≈ 1/100
- Metals ≈ 10^{-4} – 10^{-2}

Glass, diamond: fracture, close to the ideal strength
Ceramics: fracture, around 10 times lower than ideal
Metals: yield, at a stress *1000 times* weaker than ideal

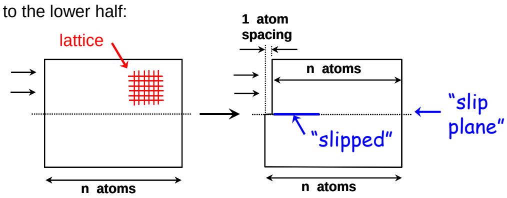
Plastic yielding therefore exploits another mechanism, enabling deformation:
- at much lower stresses than the ideal strength
- with the benefit that the material remains intact

The key to this behaviour is the *dislocation*.

6.1.2 Dislocations

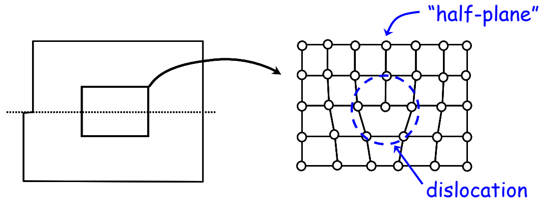
Dislocations are *line defects* in a crystal.

Consider a block of perfect material, with the atoms in a cubic lattice. Displace the top half of the block, on one side only, by one atomic spacing relative to the lower half:

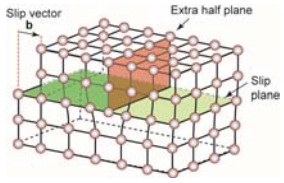


To accommodate this displacement:
- part of the interface between the blocks has *slipped*, and part has not
- the top block contains an extra *half-plane* of atoms

The extra half-plane is found at the *boundary between slipped and unslipped regions* – the crystal defect at this point is called a *dislocation*:

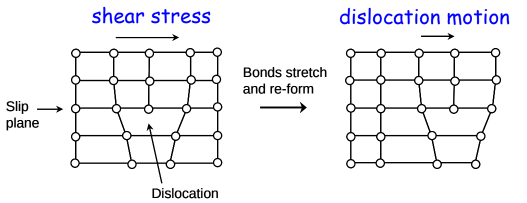


3D view:



Dislocation motion

Dislocations move by the action of *shear stress* parallel to the slip plane:

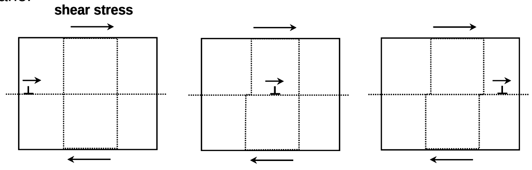


Note that when a *dislocation* moves:

- no *atom* moves more than a fraction of the atomic spacing
- the *adjacent* set of atoms become the "half-plane"

Consider a dislocation moving right through a block of material.

This gives a *net displacement* between material above and below the slip plane:



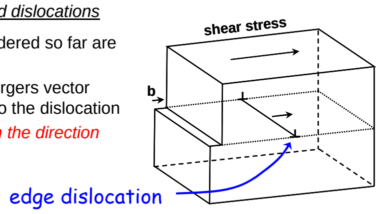
Slip step produced by the passage of one dislocation is the *Burgers vector* **b**.

Dislocations enable *incremental slip* by extending a few bonds at a time, which is why the stress required is so much less than the ideal strength.

Edge, screw and mixed dislocations

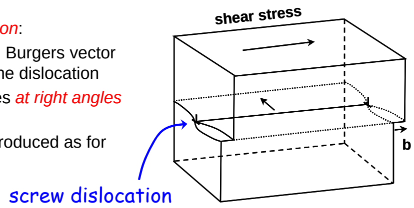
The dislocations considered so far are *edge dislocations*:

- shear stress and Burgers vector both *at right angles* to the dislocation
- dislocation moves *in the direction* of the stress



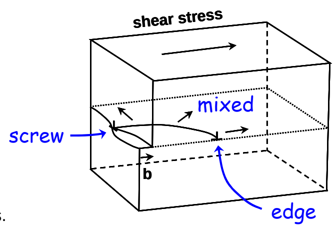
In a *screw dislocation*:

- shear stress and Burgers vector both *parallel* to the dislocation
- dislocation moves *at right angles* to the stress
- same slip step produced as for edge dislocation



More generally dislocations are *mixed*:

- *curved*, and varying between *pure edge* and *pure screw*
- move in a direction normal to the curve under the action of a shear stress (curved sections expand)
- net effect remains a slip step in the direction of the shear stress.

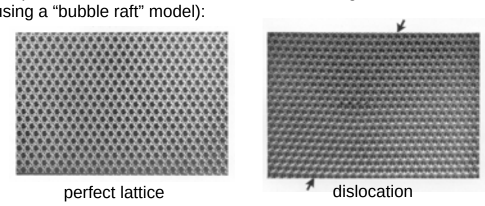


Dislocations in hexagonal lattices

Cubic lattices are convenient for visualising the idea of a dislocation.

Dislocations are essentially the same in *hexagonal lattices* (FCC, HCP), but it is harder to visualise where the atoms are.

The pictures illustrate a dislocation in a 2D hexagonal lattice (using a "bubble raft" model):



view along the arrows

perfect lattice　　dislocation

Notes:

- disruption of the lattice extends only a few atoms from the dislocation "core"
- easier to forget about the atoms and simply think of dislocations as line defects "gliding" across slip planes under the action of imposed shear stress

Incremental slip → macroscopic plastic strain

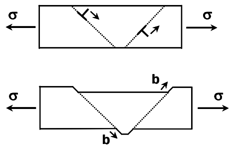
A dislocation crossing a lattice leads to an incremental slip step (in shear) of the order of one atomic spacing.

How does this enable plastic strains of 0.1–10% or more?

Two key aspects:

- crystals contain *very many dislocations*, with many different planes on which they can glide.
- in (virtually) any stress state, *shear stresses* exist to move dislocations (recall the off-axis shear stress noted in uniaxial tension).

Consider a crystal loaded in tension, with two dislocations crossing at 45º:



Net effect: crystal becomes longer and thinner by a small increment.

Replicating this increment x 1000s of dislocations on multiple slip planes produces continuum bulk plasticity.

This also shows why plastic deformation occurs *at constant volume* – blocks of material slip past one another but the crystal packing is unaffected.

6.1.3 Forces on dislocations

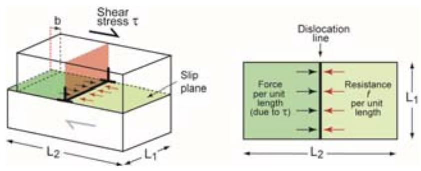
Dislocation resistance per unit length

Shear stresses apply a force (per unit length) to dislocations.

Crystals resist dislocation motion with a *resistance per unit length, f*.

The dislocation moves when this force equals the resistance.

To relate τ to f : consider the work done by τ as the dislocation moves.



For the block of material shown:

- force applied by the shear stress = $\tau(L_1 L_2)$
- when dislocation moves a distance L_2, force due to stress moves b, so the work done = $\tau(L_1 L_2)b$
- the resistance force on the length L_1: $f L_1$
- this force is moved a distance L_2, so the work done = $f L_1 L_2$
- equating work done, force (per unit length) due to shear stress: $\tau b = f$

(equally valid for edge, screw & mixed)

Intrinsic resistance to dislocation motion

The *intrinsic lattice resistance* to dislocation motion comes from additional *bond stretching* as the dislocation moves each Burgers vector step.

This resistance depends on the type of bonding:

- *Technical ceramics, diamond*: covalent bonds ⇒ high intrinsic resistance: high hardness
- *Metals*: metallic bonds ⇒ low intrinsic resistance: annealed *pure metals* are soft.

Metallic alloys are *much stronger* than pure metals: this strength is obtained by providing additional *obstacles* to dislocation motion (see below).

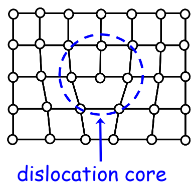
Dislocation energy per unit length – the "line tension"

Atoms around a dislocation are displaced from their equilibrium spacing, and thus have a *higher energy*.

The *energy (per unit length)* can be calculated from the *elastic stress-strain field* around the *dislocation core*:

The result is: $T \approx Gb^2/2$

(G = shear modulus; b = Burgers vector)



Effects of dislocation energy/unit length:

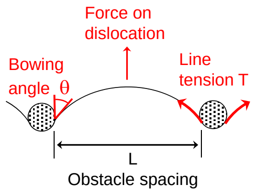
- dislocations store *elastic energy* in the lattice: this controls the response in *heat treatment* of deformed metals (e.g. *recrystallisation* – IB Materials).
- dislocations try to be as short as possible – i.e. as if they are under *tension*; *energy per unit length* is referred to as the *line tension*.
- line tension governs how dislocations interact with *obstacles*.

<u>*Dislocation pinning*</u>

When a gliding dislocation meets obstacles in its slip plane:

- it is *pinned* by the obstacles, and is forced to *bow out* between them, increasing the *resistance per unit length*
- an *additional shear stress* $\Delta\tau$ is needed to overcome this resistance

As the dislocation bows out, it applies a force to the obstacle (via the line tension):



- force on obstacle = $2\,T\cos\theta$
- dislocation escapes when either:
  - force > obstacle strength ($\theta > 0^o$)
  - dislocation forms a semi-circle ($\theta = 0^o$)

Weak obstacles: $\theta > 0^o$:
⇒ resistance force < 2T

Strong obstacles: $\theta = 0^o$:
⇒ maximum resistance force = 2T

Shear stress to overcome obstacles:

For projected length L of dislocation between obstacles, additional force due to shear stress $\Delta\tau$: $\Delta\tau\, b\, L$

Hence shear stress needed to overcome obstacles: $\Delta\tau = c\,T / b\,L$
(where c = 2: strong; c < 2: weak)

Since $T \approx Gb^2/2$: $\Delta\tau \approx \alpha\, G\, b/L \quad (\alpha \le 1)$

This is a *key result*: the *contribution to the yield stress due to dislocation pinning* depends directly on:

- G: elastic shear modulus ← fixed parameters
- b: Burgers vector (atomic spacing) ←
- L: obstacle spacing ← can be manipulated by composition and processing
- α: obstacle strength ←

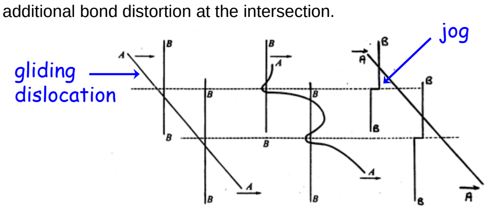
Metals and alloys use several methods to *pin dislocations*:

- other dislocations: *work hardening*
- solute atoms: *solid solution hardening*
- particles of another solid (e.g. a compound): *precipitation hardening*

6.2 Manipulating Properties II: Strength of Metals and Alloys

6.2.1 Work Hardening

Gliding dislocations on different slip planes interact: pinning occurs due to the additional bond distortion at the intersection.



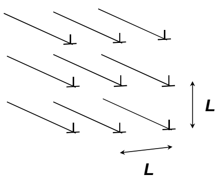
The gliding dislocation (A) bows out until the pinning point gives way, creating a *jog* in the pinning dislocation (B). Jogs then reduce the mobility of the other dislocations (B).

Strength contribution:

- additional shear stress from dislocation pinning $\propto 1/L$
- spacing *L* depends on the *dislocation density*, ρ_d
  i.e. *total dislocation length per unit volume* (units: m/m^3, or m^{-2}).

Dislocation density rises with strain – reducing the spacing, *L*, and increasing the resistance – this is called *work hardening*.

To estimate *dislocation spacing*, assume dislocations form a parallel array on a square grid, $L \times L$:



For *unit length* of dislocation:
- area per dislocation = L^2
- volume per dislocation = L^2

This is the *reciprocal* of dislocation density, ρ_d

Hence: $L = 1/\sqrt{\rho_d}$

Additional shear stress from dislocation pinning: $\Delta\tau_{wh} \propto Gb/L \propto Gb\sqrt{\rho_d}$

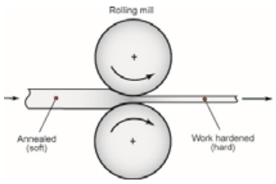
Typical microstructural data:

annealed: 10^{11} m/m^3 *work hardened*: 10^{15} m/m^3 (10^6 km/cm^3)

⇒ dislocation spacing (work hardened): $L = 1/\sqrt{10^{15}}$ m $= 32$nm

(cf. atomic diameter ≈ 0.2 nm)

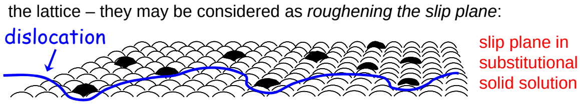
Hence alloys may be hardened by *deformation processing* (e.g. rolling, wire drawing), to increase the dislocation density while shaping the product.



6.2.2 Solid Solution Hardening

Most mixtures of metal + other elements form *solid solutions*, sometimes over wide ranges of composition.

Solute atoms have a different size and local bonding to the host atoms in the lattice – they may be considered as *roughening the slip plane*:



Interstitial solid solutions also provide hardening, by displacing host atoms from their equilibrium positions – i.e. a similar effect on the slip plane.

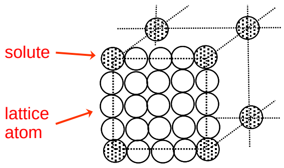
Solid solutions provide a *weak obstacle* to dislocations, which bow out until the line tension pulls the dislocation past the solute atom.

Casting is used to mix elements together in the liquid state, enabling solid solutions to be manufactured.

Strength contribution:
- additional shear stress from dislocation pinning $\propto 1/L$
- spacing L of solute atoms scales with *solute concentration C* as $1/C^{1/2}$.

Additional shear stress from solid solution: $\Delta\tau_{ss} \propto Gb/L \propto C^{1/2}$

Estimate of solute spacing in a *solid solution:*
- consider the cubic lattice shown
- solute atoms regularly spaced 4 atoms apart:



Atomic fraction of solute = 1/64 (1.6%)

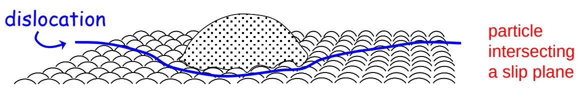
(typical values for alloys, 1-5%)

Spacing of solute atoms ≈ 0.8 nm

6.2.3 Precipitation Hardening

Alloying elements also form *compounds*. When distributed as small particles within a lattice, they provide pinning points for dislocations.

Particles may be introduced in various ways (see below) – but the hardening is referred to generally as *precipitation hardening*.



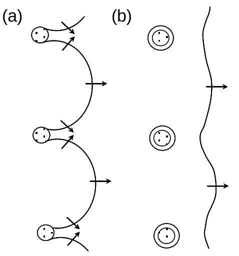
Particles provide *strong obstacles*: the dislocation cannot pass over them, and (usually) the precipitate lattice is unrelated to the surrounding lattice.

Mechanism of precipitation hardening

Maximum shear stress required to pass particles is when the dislocation bows out into a semi-circle (from above: τ b L = 2T = Gb²).

Additional shear stress from precipitation hardening:

$$\Delta\tau_{ppt} = Gb/L$$



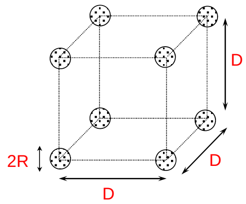
(a) the dislocation escapes by the linking of two adjacent bowing dislocations.

(b) a *dislocation loop* is left round the particle.

Estimate of particle spacing in precipitation hardening:

The *particle spacing* is determined by their *size* and *volume fraction*.

Assume a cubic array of *particles* of *radius R*, and centre-to-centre *spacing D*:



(NB: these are *particles*, not *atoms*).

Each particle also occupies the *centre* of a cube of side D.

Hence the volume fraction *f* of particles:

$$f = (4/3)\,\pi\,R^3/D^3$$

e.g. for typical volume fraction $f \approx 5\%$, and particle radius $R \approx 25$ nm:

$$D \approx 110 \text{ nm}$$

Minimum gap between particles:

$$L = D - 2R \approx 60 \text{ nm}$$

(cf. dislocation spacing ≈ 30nm; solute spacing ≈ 1nm)

6.2.4 Yield in Polycrystals

So far: dislocation behaviour relates to dislocations in a *single crystal*, under the action of a *shear stress parallel to the slip plane*.

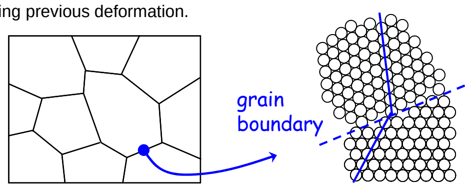
Metals are *polycrystalline*, so how does this affect dislocations?

Grains, and grain boundaries

Grains are produced in solid metals as a result of *processing* (IB Materials):

- *casting*: solidification occurs by *nucleation and growth* of tiny solid crystals – these grow randomly until they impinge, forming grains;
- *recrystallisation*, grains re-form in the solid-state, by heat treatment following previous deformation.

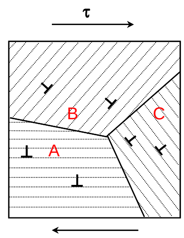
typical grain size ≈ 100 μm



2D section through grains

change in lattice orientation

Dislocation motion in a polycrystal



- under a *remote shear stress* τ, the slip planes in different grains will vary in their alignment with the stress
- dislocations move first in grains which are favourably oriented (A)
- *yield occurs progressively* throughout all the grains (B,C), at a higher remote shear stress

Shear stress needed to move dislocations: τ_y (*acting parallel to a slip plane*).

The corresponding *remote* shear stress is typically: $\tau = 3/2\ \tau_y$

This is called the *shear yield stress, k*.

To relate the *yield stress* σ_y to the *shear yield stress k*, note that:

- a uniaxial stress gives maximum shear stress at 45º to the uniaxial axis
- magnitude of the shear stress is ½ the uniaxial stress

Hence: $\sigma_y = 2k = 3\tau_y$ i.e. all previous hardening mechanisms directly increase the macroscopic yield stress.

Footnote: grain boundary hardening – the effect of grain size

The lattice orientation changes at a grain boundary. As a result:

- dislocations cannot slip directly from grain to grain
- dislocation pile-ups occur at the boundaries
- additional stress from pile-up nucleates dislocations in the adjoining grain

The finer the *grain size d*, the more often boundaries obstruct dislocations.

Grain boundary hardening given by *Hall-Petch relationship*: $(\Delta\sigma_y)_{gb} \propto 1/\sqrt{d}$

(Note: this is a weak hardening mechanism – grain boundaries are *much* further apart than dislocations, solute or precipitates. It is useful as a strengthening mechanism for *pure metals* or *dilute alloys*).

6.2.5 Comparison of hardening mechanisms

Yield stress data for work hardened alloys

Pure Cu, σ_y: 50-60 MPa — $(\Delta\sigma_y)_{wh} \approx$ few MPa

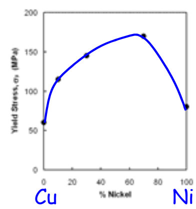
Cold-drawn Cu, σ_y: 180-350 MPa — $(\Delta\sigma_y)_{wh} \approx$ 120-300 MPa

This factor of ≈100 in $(\Delta\sigma_y)_{wh}$ corresponds to a factor of 100² = 10,000 in dislocation density.

Yield stress data for solid solution hardened alloys

CES data for σ_y of *Cu-Ni alloys*: solid solution from 100% Cu to 100% Ni.

Alloy	σ_y (MPa)
Pure Cu	60
Cu – 10% Ni	115
Cu – 30% Ni	145
Cu – 70% Ni	170
Pure Ni	80



NOT a rule of mixtures

Yield stress data for precipitation hardened alloys

Pure Al: 25 MPa *High strength aerospace Al alloy*: 500 MPa

Pure Fe: 110 MPa *Quenched/tempered high alloy (tool) steel*: 2000 MPa

What particle spacing (and size) gives useful precipitation hardening?

Example: what particle spacing in Al alloy gives a *yield stress increment* $(\Delta\sigma_y)_{ppt}$ of 400 MPa?

Recall for precipitation hardening:

- increment in shear stress to bow dislocations: $\Delta\tau_y \approx G\,b\,/\,L$
- yield stress increment is: $\Delta\sigma_y \approx 3\,\Delta\tau_y$

For aluminium: shear modulus G = 26 GPa, Burgers vector b = 0.286 nm.

Hence: $L \approx 3\,G\,b\,/\,(\Delta\sigma_y)_{ppt} \approx 55$ nm

(Close to previous estimate for a volume fraction 5% of spherical particles of radius 25nm).

Consequences: processing for precipitation hardening

A few % of particles around 25nm radius gives a useful strength increment (e.g. 400MPa in Al).

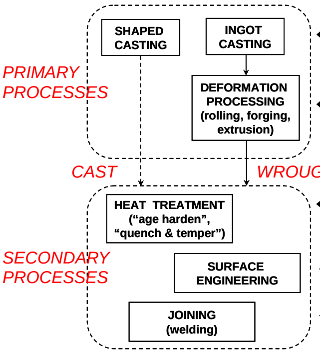
It is practically very *difficult* to manufacture solid particles *this small*, and to mix them into a melt before *casting*.

The main practical manufacturing route is to use *heat treatment* in the solid state, forming fine *precipitates* (from a solid solution) – hence the name "precipitation hardening":

- controlled *temperature-time histories* offer a versatile route to controlling precipitate structure, size and volume fraction (IB Materials)
- practical precipitates vary in size from clusters of 10 or so atoms, to compounds containing 10^6 or more atoms (i.e. diameters 0.5 – 200 nm)

6.2.6 Overview: alloy processing for strength

- *many processes* for making metal components (often multi-stage)
- alloy *composition* & *process route* determine *microstructure*, and thus σ_y



- fix composition (solute)
- form initial grain size
- work hardening (increase ρ_d)
- precipitation hardening
- annealing (reduce ρ_d, modify grain size)
- harden surface
- heat alters microstructure and properties – may lead to failure

Examples of alloys, applications and hardening mechanisms

Alloy	Typical uses	Work hardening	Solid solution hardening	Precipitation hardening
Pure Al	Foil	XXX		
Pure Cu	Wire	XXX		
Cast Al, Mg	Automotive parts		XXX	X
Bronze (Cu-Sn), Brass (Cu-Zn)		X	XXX	X
Non-heat-treatable wrought Al	Ships, cans, structures	XXX	XXX	
Heat-treatable wrought Al	Aircraft, structures	X	X	XXX
Low carbon steels	Car bodies, ships, structures, cans	XXX	XXX	
Low alloy steels	Automotive parts, tools	X	X	XXX
Stainless steels	Cutlery, pressure vessels	XXX	XXX	
Cast Ni alloys	Jet engine turbines		XXX	XXX

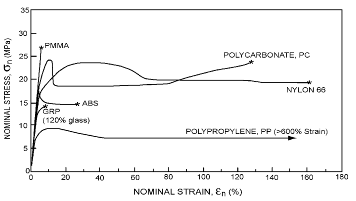
XXX: routinely used; X: sometimes used

6.3 Failure of Polymers

Polymer strength is determined by:

- molecular architecture and bonding
- the ability of the chain molecules to unravel and slide (no real equivalent to the dislocation)
- temperature, relative to the glass transition, and the strain-rate

Selected $\sigma-\varepsilon$ curves for polymers, at room temperature (from *Materials Databook*).

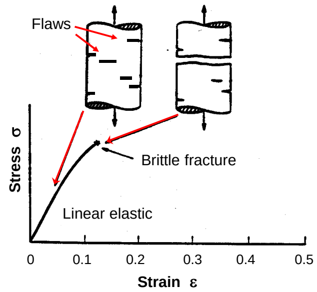


Amorphous Thermoplastics

(1) $T < 0.8\,T_g$: *elastic-brittle*

- chain sliding limited
- brittle fracture from inherent flaws in the material
- little or no ductility

e.g. PMMA (tension)

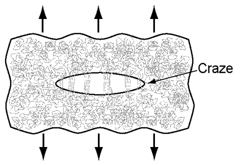


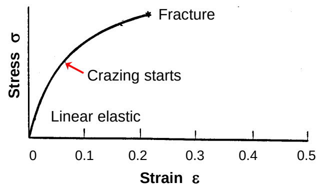
(2) 0.8 T_g < T < 1.2 T_g: *elastic-plastic*

- chain mobility increases around T_g as van der Waals bonds melt
- yielding takes place by *crazing, shear yielding* or *cold drawing*.

Crazing:

Microcracks open in tension, bridged by stiff fibres of material with aligned molecules, preventing immediate fracture.

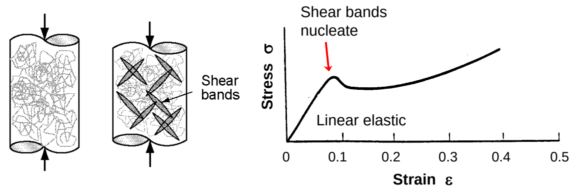




e.g. PMMA (compression)
polycarbonate (tension/compression)

Shear yielding:

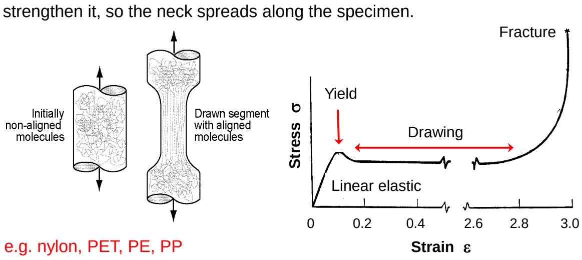
Shear bands form, and are stabilised by alignment of molecules; multiple bands form, giving greater ductility.



Cold drawing

Polymers which do not craze can often be cold drawn.

Necking occurs, but the *neck is stable*: the molecules align in the neck and strengthen it, so the neck spreads along the specimen.

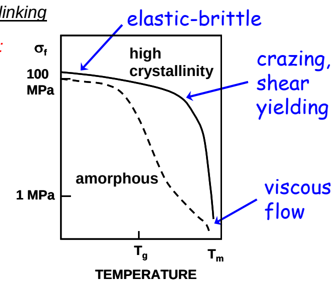


e.g. nylon, PET, PE, PP

Effect of crystallinity and cross-linking

Semi-crystalline thermoplastics:

- strength follows a similar pattern to Young's modulus
- above T_g crystalline regions resist deformation: strength increases with crystallinity

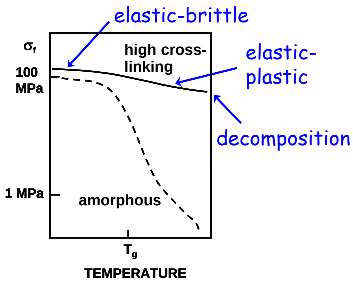


Elastomers:

- *elastic-brittle* below T_g, but very *large elastic strains* to failure above T_g
- fail catastrophically with little or no ductility: elastic strain is recovered
- it is confusing (but strictly true) that the *elongation to failure is zero*

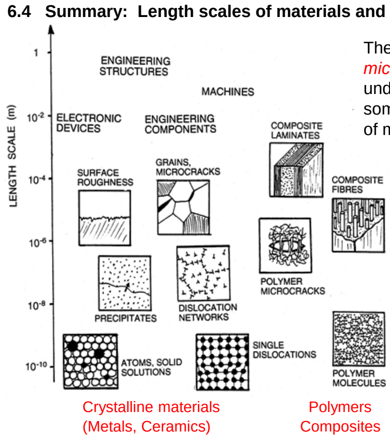
Thermosets:

- also follow the pattern of Young's modulus
- little effect of T_g: slow fall in strength until the material decomposes
- above T_g limited shear yielding may occur (desirable in epoxy resins used as the matrix in fibre composites)



6.4 Summary: Length scales of materials and microstructures

The figure summarises the *microstructural features* that underpin the structural (and some functional) properties of materials in the IA course.



7. Strength-limited Component Design

Selection of materials was introduced for *stiffness-limited* design, at *minimum weight or cost*.

Many structural components are also *strength-limited*: this can be analysed following the same methodology:

(1) identify *objective* (e.g. minimum mass or cost)

(2) identify *functional constraint* (i.e. must not fail: $\sigma_{max} < \sigma_f$)

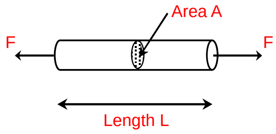
(3) examine *geometrical constraints* (fixed dimensions, free variables)

7.1 Selection of light, strong materials

Example: Light, *strong* tensile tie

A tensile tie of specified length *L* is required to carry a load *F*, *without failure*.

The tie has a uniform prismatic cross-section, but its area *A* may be varied.



Step 1: *Objective*: minimum mass $m = \rho L A$

Step 2: *Functional constraint*: must not fail, $\sigma_{max} < \sigma_f$ $\sigma = F/A = \sigma_f$

Step 3: *Geometric constraint*: fixed L, free variable A

Hence strength constraint becomes: $F = A\,\sigma_f$ = constant

Eliminate the free variable A in the objective equation:

Mass m = $(L\,F)\ (\rho/\sigma_f)$ i.e. mass $\propto (\rho/\sigma_f)$

The mass is minimised by maximising the *performance index*: (σ_f/ρ)

This is the *specific strength*. As with E/ρ, it is commonly used to compare materials, but is not always the optimum combination.

For minimum *material cost*, the performance index is modified as before:

$Cost = \rho\, L\, A\, C_m \Rightarrow$ maximise $(\sigma_f/\rho\, C_m)$

Light, strong components in bending

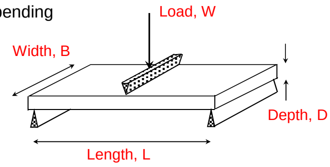
Shaping the cross-section improves *stiffness* in bending, and also reduces the *maximum stress* generated by a given bending moment (IA Structures).

The effect of section shape on material selection is considered later.

To investigate the effect of strength-limited design for *bending*, as opposed to tension, consider material selection for a *light, strong panel.*

Example: Light, strong panel in bending

- specified span L, width B
- carry load *W* in 3-point bending, *without failure*
- rectangular cross-section, depth D may be varied



Following the same procedure as before:

Objective: minimum mass

Functional constraint: Set max. stress = failure stress: $\dfrac{\sigma_{max}}{y_{max}} = \dfrac{M}{I}$ (where $I = BD^3/12$)

Geometric constraint: length L, width B fixed; free variable *D*

Full analysis in Examples Paper 4 – resulting performance index is: $(\sigma_f^{1/2}/\rho)$

Material selection for minimum mass

(1) On *Strength – Density property chart*:

Take logs as before, and re-arrange into form $y = m\,x + c$:

(σ_f/ρ) = constant: lines of slope 1

$(\sigma_f^{1/2}/\rho)$ = constant: lines of slope 2

(2) Apply *secondary constraints* (as before):

Avoid brittle materials (ceramics, glass)

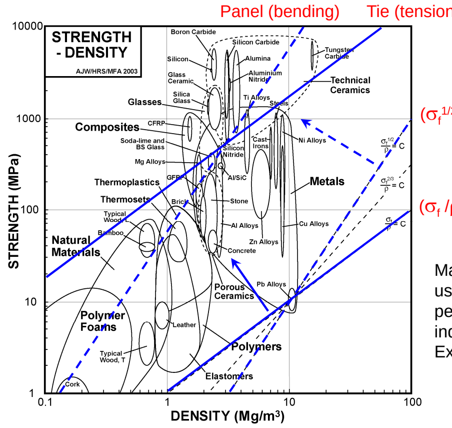
Upper limit on cost/kg

Environmental resistance requirements

Manufacturing limits

Size limits – e.g. in tension: $A\,\sigma_f$ = constant:

upper limit on $A \Rightarrow$ lower limit on σ_f

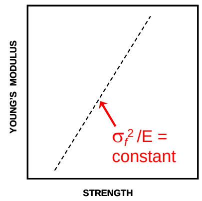


Material selection using these performance indices: see Examples Paper 4.

7.2 Case studies in strength-limited design

(1) *Materials for springs*

Maximum elastic stored energy per unit volume was shown earlier: $\sigma_f^2/2E$



Property chart in Materials Databook, or in CES (Examples Paper 4).

Take care to:

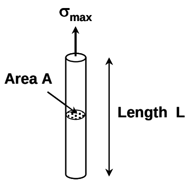
(a) apply correct slope for index

(b) move line in correct direction to optimise the selection.

(2) *Failure under self-weight: suspended cables*

A cable of uniform cross-sectional area A hangs vertically under its own weight.

Find a performance index that *maximises the length* that can hang *without failure*.



Objective: maximum length, L

Constraint: $\sigma_{max} < \sigma_f$

Free variable: area, A

Max. stress, σ_{max} = weight/area = $\rho\,(AL)\,g\,/\,A$
$= \rho\, L\, g$

Hence length at failure = $\sigma_f/\rho\, g$

For longest cable at failure – maximise σ_f/ρ

Notes:

- for a cable of *given length*, the analysis sets a *lower limit* on (σ_f/ρ)
- cables suspended across a span with a shallow dip (as in IA Structures) may be analysed in the same way

7.3 Effect of shape on material selection for lightweight design

Section shape is used to improve the efficiency of components and structures loaded in bending, e.g. I-beams:

(The same applies in torsion – twisting – e.g. hollow tubes).

To include shape in material selection, we need to:

- quantify the efficiency of section shape
- consider both stiffness and strength

Shape efficiency in bending: stiffness

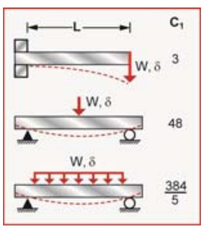
Bending *stiffness* is governed by the *flexural rigidity,* EI (cf. IA Structures):

$$\text{Stiffness, } S = \frac{W}{\delta} = \frac{C_1 EI}{L^3}$$

where I = second moment of area

$$I = \int y^2\, dA$$

and C_1 depends on the loading geometry.



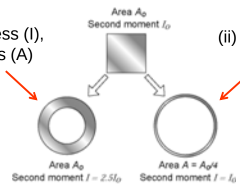
Note that:

- *stiffness*, S $\propto$ second moment of area, I
- *mass* (per unit length), m/L $\propto$ area, A

Shaping a section may be considered to improve efficiency in two ways:

(i) increased stiffness (I), at constant mass (A)

(ii) reduced mass (A), at constant stiffness (I)



Consider case (i): constant area (and mass/length):

Define *shape factor, for stiffness in bending*, $\Phi_e = \dfrac{\text{I for shaped section}}{\text{I for reference shape}}$

A simple reference shape is a solid square section:

Area, $A = b^2$ Second moment, $I_o = \dfrac{b^4}{12} = \dfrac{A^2}{12}$

Hence shape factor for elastic bending stiffness: $\Phi_e = \dfrac{12\,I}{A^2}$

(cf. a dimensionless group)

N.B. There are *physical limits* to the magnitude of the shape factor: this leads to a *maximum shape factor* for each material (see below).

Case (ii) is more relevant to material selection: minimum mass for a given stiffness.

Recall how to derive a performance index for minimum mass, in bending:

Objective: minimum mass, $m = \rho L A$

Functional constraint: bending stiffness $S = \dfrac{W}{\delta} = \dfrac{C_1 EI}{L^3}$

Geometric constraints: *L* fixed; *shape* and *area* now free variables

The stiffness constraint is: $\dfrac{(W/\delta)\,L^3}{C_1} = EI = \text{constant}$

Substituting for I, using the shape factor, Φ_e: $\dfrac{(W/\delta)\,L^3}{C_1} = \dfrac{E\,\Phi_e A^2}{12}$

Hence area $A \propto 1/(E\,\Phi_e)^{1/2}$

Substituting into objective equation: mass, $m \propto \dfrac{\rho}{(E\,\Phi_e)^{1/2}}$

Hence for minimum mass, maximise performance index: $\dfrac{(E\,\Phi_e)^{1/2}}{\rho}$

For the same *stiffness*, shaping a section reduces the mass (relative to a solid square section) by a factor of $1/(\Phi_e)^{1/2}$.

Maximum shape factor: stiffness

The *maximum shape factor* depends on the *physical limits* on section thickness due to:

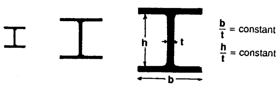
- the capabilities of *manufacturing processes*
- *buckling failure* of thin-walled sections

Key point: different materials can be shaped to a different extent.

Material	*Typical maximum shape factor,* Φ_e	*Typical mass ratio by shaping,* $1/(\Phi_e)^{1/2}$
Steels	64	1/8
Al alloys	49	1/7
Fibre Composites	36	1/6
Wood	9	1/3

Numerical values for performance index, with and without shape:

For *constant* Φ_e, the shape is fixed:



i.e. as area varies, the dimensions remain in constant proportion

In this case the performance index becomes: $\dfrac{E^{1/2}}{\rho}$

Material	Index with fixed shape, $E^{1/2}/\rho$	Index including max. shape factor, $(E\,\Phi_e)^{1/2}/\rho$
Steels	1.86	14.9
Al alloys	3.10	21.7
CFRP	6.25	37.5
Wood	4.84	14.5

Notes:

- Composites lose some of their performance advantage over metals
- Wood falls behind in applications which can exploit shape

Shape factor for bending strength

Similar arguments apply to quantify the effect of shape on *strength*.

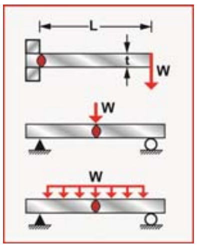
Bending *strength* is governed by the maximum moment, M, and the corresponding maximum stress, σ (cf. IA Structures):

$$\frac{\sigma_{max}}{y_{max}} = \frac{M}{I}$$

where I = second moment of area

Failure moment: $M_f = \left(\dfrac{I}{y_{max}}\right)\sigma_f = Z_e\,\sigma_f$

where Z_e is the *elastic section modulus*.



Define *shape factor*, *for strength in bending*, $\Phi_f = \dfrac{Z_e \text{ for shaped section}}{Z_e \text{ for reference shape}}$

Using the same reference shape of a solid square section: $\Phi_f = \dfrac{6\,Z_e}{A^{3/2}}$

For minimum mass, maximise performance index: $(\sigma_f\,\Phi_f)^{2/3}/\rho$

(derivations on Examples Paper 4).

Notes:

- for the same *strength*, shaping reduces the mass (relative to a solid square section) by a factor of $1/(\Phi_f)^{2/3}$.
- for *constant* Φ_f (fixed shape), the index becomes $\sigma^{2/3}/\rho$.

Maximum shape factor: strength

The same physical limits on section thickness determine the *maximum shape factor* for strength, for each material class:

Material	*Typical maximum shape factor,* Φ_f	*Typical mass ratio by shaping,* $1/(\Phi_f)^{2/3}$
Steels	13	0.18
Al alloys	10	0.22
Fibre Composites	9	0.23
Wood	3	0.48

Notes:

- Shaping has a smaller influence on strength than on stiffness (because increasing I is partly achieved by increasing y_{max})
- Metals again catch up a little with composites; wood falls behind.

Summary: solving problems with shape

The differences in shape factor between materials are of comparable magnitude to the differences in modulus and strength:

- *shape is significant in material selection for bending applications*.

Hence if area and shape can both be varied:

- *either*, use performance indices including shape factor
- *or*, use performance indices *without* shape, but comment on likely effect of shape (metals > composites > wood)

Case study: *Plastic for lightweight bicycles ?*

Bike frames are limited by *both stiffness and strength*, and may be optimised for low mass or low cost, depending on the market.

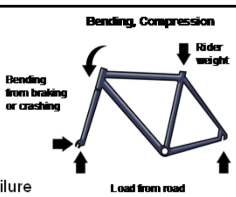
Full analysis of problems with *more than* one functional constraint is discussed below.

A *preliminary analysis* may be conducted to consider the question: would a plastic bicycle be lightweight?

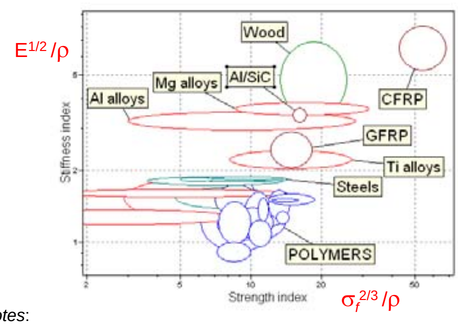
Assume the following:

- frame loading is dominated by *bending*
- *shape is fixed* (e.g. tubes of given radius:thickness ratio), *size may vary*
- recall the relevant performance indices to maximise for *minimum mass* are:

$\dfrac{E^{1/2}}{\rho}$ for given stiffness, $\dfrac{\sigma_f^{2/3}}{\rho}$ to avoid failure



Plot these indices against one another in CES.



Notes:

- strong competition between Al, Mg, Ti alloys, Al-SiC and GFRP
- steels do not perform well for low weight; CFRP is outstanding
- polymers cannot compete, particularly on stiffness
- wood performs well, but cannot in practice be made into thin-walled tubes

7.4 Material selection with multiple constraints

In earlier examples of lightweight design (with fixed shape):

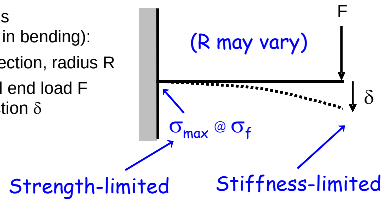
- *objective*: minimum mass

either, stiffness-limited → functional constraint: given stiffness

or, strength-limited → functional constraint : avoid failure

Performance index analysis
(e.g. lightweight cantilever in bending):

- assume solid circular section, radius R
- fixed length L, specified end load F and allowable tip deflection δ
- OR, must not fail



Objective: minimum mass, $m = \rho L \pi R^2$

Stiffness constraint:

$$\frac{F}{\delta} = \frac{3EI}{L^3} = \frac{3E\pi R^4}{4L^3}$$

Eliminating R:

$$m_\delta = \rho \pi L \left(\frac{4FL^3}{3\pi\delta E}\right)^{1/2}$$

$$m_\delta = \left(\frac{4\pi F L^5}{3\delta}\right)^{1/2} \left(\frac{\rho}{E^{1/2}}\right)$$

Strength constraint:

$$\frac{\sigma_f}{R} = \frac{FL}{I} = \frac{4FL}{\pi R^4}$$

Eliminating R:

$$m_\sigma = \rho \pi L \left(\frac{4FL}{\pi \sigma_f}\right)^{2/3}$$

$$m_\sigma = \left(\pi L^5\right)^{1/3} (4F)^{2/3} \left(\frac{\rho}{\sigma_f^{2/3}}\right)$$

If *one constraint* (stiffness *or* strength):

- minimise mass → maximise appropriate index ($E^{1/2}/\rho$ or $\sigma_f^{2/3}/\rho$)
- do not need values for F, L, δ

If *both constraints* apply:

- *limiting mass is the higher of m_δ and m_σ* (to guarantee both are met)
- must evaluate actual masses ⇒ need values for F, L, δ
- lightest material: the lowest of the limiting mass values

Example: cantilever with stiffness and strength constraint

Fixed length *L* = 0.5m, end load *F* = 500N

Allowable deflection δ = 50mm; must not fail (max. stress < σ_f)

	E GPa	*ρ* kg/m³	σ_f MPa	m_δ kg	m_σ kg	Design-limiting constraint
CFRP	120	1500	600	**0.16**	0.15	Stiffness
Ti alloy	120	4500	700	**0.47**	0.42	Stiffness
Al alloy	70	2700	400	**0.37**	0.36	Stiffness
Alloy steel	210	7800	600	0.62	**0.80**	Strength
Nylon	3	1100	100	**0.73**	0.37	Stiffness
Wood	12	600	70	0.20	**0.26**	Strength

Result: CFRP is the lightest material

Notes:

- in this example, CFRP was the lightest material for both constraints
- this is not always the case: the best material may not in fact be the lightest on *either* criterion

<u>Further refinements</u>

Minimum cost

- same analysis, with each limiting mass × cost/kg to convert to cost
- same design-limiting criterion will apply for each material

	m kg	C_m £/kg	*Cost* £
CFRP	0.16	60	**9.6**
Ti alloy	0.47	40	**18.8**
Al alloy	0.37	2	**0.74**
Alloy steel	0.80	1	**0.80**
Nylon	0.73	4	**2.92**
Wood	0.26	2	**0.52**

Result:
Wood is the cheapest material

Size limits

- given limiting mass for each material, back-substitute into objective equation to find actual size required (e.g. radius R in example)

Secondary constraints

- as in earlier examples: comment on toughness, corrosion, manufacturing, joining etc.