
6.1 Atomic basis of Plasticity in Crystalline Materials

Recall that in crystalline materials, the key features of atomic packing are:

- atoms/ions pack together as hard spheres
- they pack in planes, which stack to form the lattice
- lattices are close-packed (FCC, HCP), or nearly so (BCC)
- straight lines of touching atoms form close-packed directions.

The atomic bonding is strong and primary: metallic, ionic or covalent.

- how is this achieved at the atomic level?
- can the behaviour be manipulated to increase material strength?

Elastic deformation displaces atoms by a fraction of their equilibrium spacing.

Plastic deformation involves relative movement of material over very large multiples of the atomic spacing.

The issues therefore are:
- how is this achieved at the atomic level?
- can the behaviour be manipulated to increase material strength?

6.1.1 Ideal Strength of Crystalline Material

Estimate of upper limit on strength from atomic force-distance curve:

- bonds rupture at the dissociation separation, of order $1.1 \times r_0$, i.e. a strain of approx. 10%.
- force – extension curve is linear near equilibrium separation, and is shown extrapolated in the figure.
- notional linear-elastic force at separation of $1.1 r_0$ is higher than the peak force – by a factor of order 1.5.

Tensile stress needed to break all the bonds simultaneously is thus of order $1/1.5$ of a notional elastic stress at a strain of 10%:

$$\sigma_{\text{peak}} \approx \frac{1}{1.5} \times E \times 0.1 \approx \frac{E}{15}$$

This is an estimate of the ideal strength of a material.
The extra half-plane is found at the boundary between slipped and unslipped regions – the crystal defect at this point is called a dislocation:

Dislocation motion
Dislocations move by the action of shear stress parallel to the slip plane:

Note that when a dislocation moves:
• no atom moves more than a fraction of the atomic spacing
• the adjacent set of atoms become the “half-plane”

Consider a dislocation moving right through a block of material. This gives a net displacement between material above and below the slip plane:

Slip step produced by the passage of one dislocation is the Burgers vector \( b \).

Dislocations enable incremental slip by extending a few bonds at a time, which is why the stress required is so much less than the ideal strength.

More generally dislocations are mixed:
- curved, and varying between pure edge and pure screw
- move in a direction normal to the curve under the action of a shear stress (curved sections expand)
- net effect remains a slip step in the direction of the shear stress.

Dislocations in hexagonal lattices
Cubic lattices are convenient for visualising the idea of a dislocation. Dislocations are essentially the same in hexagonal lattices (FCC, HCP), but it is harder to visualise where the atoms are.

The pictures illustrate a dislocation in a 2D hexagonal lattice (using a “bubble raft” model):

Notes:
• disruption of the lattice extends only a few atoms from the dislocation “core”
• easier to forget about the atoms and simply think of dislocations as line defects “gliding” across slip planes under the action of imposed shear stress
Incremental slip \( \rightarrow \) macroscopic plastic strain

A dislocation crossing a lattice leads to an incremental slip step (in shear) of the order of one atomic spacing.

How does this enable plastic strains of 0.1–10% or more?

Two key aspects:

• crystals contain very many dislocations, with many different planes on which they can glide.
• in (virtually) any stress state, shear stresses exist to move dislocations (recall the off-axis shear stress noted in uniaxial tension).

Consider a crystal loaded in tension, with two dislocations crossing at 45°:

Net effect: crystal becomes longer and thinner by a small increment.

Replicating this increment x 1000s of dislocations on multiple slip planes produces continuum bulk plasticity.

This also shows why plastic deformation occurs at constant volume – blocks of material slip past one another but the crystal packing is unaffected.

6.1.3 Forces on dislocations

Dislocation resistance per unit length:

Shear stresses apply a force (per unit length) to dislocations. Crystals resist dislocation motion with a resistance per unit length, \( f \).

The dislocation moves when this force equals the resistance. To relate \( \tau \) to \( f \): consider the work done by \( \tau \) as the dislocation moves.

For the block of material shown:

- Force applied by the shear stress = \( \tau (L_1 L_2) \)
- When dislocation moves a distance \( L_2 \), force due to stress moves \( bh \), so the work done = \( \tau (L_1 L_2) bh \)
- The resistance force on the length \( L_1 \): \( fL_1 \)
- This force is moved a distance \( L_2 \), so the work done = \( fL_1 L_2 \)
- Equating work done, force (per unit length) due to shear stress: \( f = \frac{\tau bh}{L_2} \) (equally valid for edge, screw & mixed)

Intrinsic resistance to dislocation motion:

The intrinsic lattice resistance to dislocation motion comes from additional bond stretching as the dislocation moves each Burgers vector step.

This resistance depends on the type of bonding:

• Technical ceramics, diamond: covalent bonds \( \Rightarrow \) high intrinsic resistance: high hardness
• Metals: metallic bonds \( \Rightarrow \) low intrinsic resistance: annealed pure metals are soft.

Metallic alloys are much stronger than pure metals: this strength is obtained by providing additional obstacles to dislocation motion (see below).

Dislocation energy per unit length – the "line tension":

Atoms around a dislocation are displaced from their equilibrium spacing, and thus have a higher energy.

The energy (per unit length) can be calculated from the elastic stress-strain field around the dislocation core:

The result is: \( I = Gb^2/2 \)

\( (G = \text{shear modulus}; b = \text{Burgers vector}) \)
Effects of dislocation energy/unit length:

- dislocations store elastic energy in the lattice; this controls the response in heat treatment of deformed metals (e.g. recrystallisation – IB Materials).
- dislocations try to be as short as possible – i.e. as if they are under tension; energy per unit length is referred to as the line tension.
- line tension governs how dislocations interact with obstacles.

Dislocation pinning

When a gliding dislocation meets obstacles in its slip plane:

- it is pinned by the obstacles, and is forced to bow out between them, increasing the resistance per unit length
- an additional shear stress $\Delta \tau$ is needed to overcome this resistance

As the dislocation bows out, it applies a force to the obstacle (via the line tension):

- force on obstacle $= 2T \cos \theta$
- dislocation escapes when either:
  - force $> \text{obstacle strength} (\theta > 0^\circ)$
  - dislocation forms a semi-circle ($\theta = 0^\circ$)

Weak obstacles: $\theta > 0^\circ$

$\Rightarrow$ resistance force $< 2T$

Strong obstacles: $\theta = 0^\circ$

$\Rightarrow$ maximum resistance force $= 2T$

Shear stress to overcome obstacles:

For projected length $L$ of dislocation between obstacles, additional force due to shear stress $\Delta \tau$: $\Delta \tau = c \ T / b L$

Hence shear stress needed to overcome obstacles: $\Delta \tau = c \ T / b L$

(where $c = 2$: strong; $c < 2$: weak)

Since $T = G b^2 / 2$: $\Delta \tau = a G \ b / L$ ($a < 1$)

This is a key result: the contribution to the yield stress due to dislocation pinning depends directly on:

- $G$: elastic shear modulus
- $b$: Burgers vector (atomic spacing)
- $L$: obstacle spacing
- $\alpha$: obstacle strength

Metals and alloys use several methods to pin dislocations:

- other dislocations: work hardening
- solute atoms: solid solution hardening
- particles of another solid (e.g. a compound): precipitation hardening

6.2 Manipulating Properties II: Strength of Metals and Alloys

6.2.1 Work Hardening

Gliding dislocations on different slip planes interact: pinning occurs due to the additional bond distortion at the intersection.

The gliding dislocation (A) bows out until the pinning point gives way, creating a jog in the pinning dislocation (B). Jogs then reduce the mobility of the other dislocations (B).
To estimate dislocation spacing, assume dislocations form a parallel array on a square grid, \( L \times L \):

For unit length of dislocation:
- area per dislocation = \( L^2 \)
- volume per dislocation = \( L^3 \)

This is the reciprocal of dislocation density, \( \rho_d \)

Hence: \( L = 1/\sqrt{\rho_d} \)

Additional shear stress from dislocation pinning: \( \Delta \tau_{\text{dis}} = Gb/L = Gb/\sqrt{\rho_d} \)

Typical microstructural data:
- annealed: \( 10^{11} \text{ m/m}^3 \)
- work hardened: \( 10^{15} \text{ m/m}^3 \) (\( 10^6 \text{ km/cm}^3 \))

\( \Rightarrow \) dislocation spacing (work hardened): \( L = 1/10^{15} \text{ m} = 32 \text{ nm} \)

(Cf. atomic diameter \( \approx 0.2 \text{ nm} \))

Hence alloys may be hardened by deformation processing (e.g. rolling, wire drawing), to increase the dislocation density while shaping the product.

6.2.2 Solid Solution Hardening

Most mixtures of metal + other elements form solid solutions, sometimes over wide ranges of composition. Solute atoms have a different size and local bonding to the host atoms in the lattice – they may be considered as roughening the slip plane:

- slip plane in substitutional solid solution
- Interstitial solid solutions also provide hardening, by displacing host atoms from their equilibrium positions – i.e. a similar effect on the slip plane.

Solid solutions provide a weak obstacle to dislocations, which bow out until the line tension pulls the dislocation past the solute atom. Casting is used to mix elements together in the liquid state, enabling solid solutions to be manufactured.

6.2.3 Precipitation Hardening

Alloying elements also form compounds. When distributed as small particles within a lattice, they provide pinning points for dislocations.

Particles may be introduced in various ways (see below) – but the hardening is referred to generally as precipitation hardening.

Particles provide strong obstacles: the dislocation cannot pass over them, and (usually) the precipitate lattice is unrelated to the surrounding lattice.
Mechanism of precipitation hardening

Maximum shear stress required to pass particles is when the dislocation bows out into a semi-circle (from above: \( \tau = \frac{2T}{bL} = \frac{Gb^2}{2} \)).

- Additional shear stress from precipitation hardening:
  \[ \Delta \tau_{ppt} = \frac{Gb}{L} \]

- (a) the dislocation escapes by the linking of two adjacent bowing dislocations.
- (b) a dislocation loop is left round the particle.

Maximum shear stress required to pass particles is when the dislocation bows out into a semi-circle (from above: \( \tau = \frac{2T}{bL} = \frac{Gb^2}{2} \)).

6.2.4 Yield in Polycrystals

So far: dislocation behaviour relates to dislocations in a single crystal, under the action of a shear stress parallel to the slip plane.

- Grains, and grain boundaries
  - Grains are produced in solid metals as a result of processing (IB Materials):
    - casting: solidification occurs by nucleation and growth of tiny solid crystals – these grow randomly until they impinge, forming grains;
    - recrystallisation, grains re-form in the solid-state, by heat treatment following previous deformation.

- Footnote: grain boundary hardening – the effect of grain size
  - The lattice orientation changes at a grain boundary. As a result:
    - dislocations cannot slip directly from grain to grain
    - dislocation pile-ups occur at the boundaries
    - additional stress from pile-up nucleates dislocations in the adjoining grain

  The finer the grain size \( d \), the more often boundaries obstruct dislocations.

  Grain boundary hardening given by Hall-Petch relationship: \( (\Delta \sigma)_{gb} \propto 1/\sqrt{d} \)

  (Note: this is a weak hardening mechanism – grain boundaries are much further apart than dislocations, solute or precipitates. It is useful as a strengthening mechanism for pure metals or dilute alloys).

6.2.5 Comparison of hardening mechanisms

Yield stress data for work hardened alloys

- Pure Cu, \( \sigma_y \): 50-60 MPa
- Cold-drawn Cu, \( \sigma_y \): 180-350 MPa

\( (\Delta \sigma)_{wh} \) few MPa \( (\Delta \sigma)_{wh} \) 120-300 MPa

This factor of \( \approx 100 \) in \( (\Delta \sigma)_{wh} \) corresponds to a factor of \( 100^2 = 10,000 \) in dislocation density.

Yield stress data for solid solution hardened alloys

CPS data for \( \sigma_y \) of Cu-Ni alloys: solid solution from 100% Cu to 100% Ni.

<table>
<thead>
<tr>
<th>Alloy</th>
<th>( \sigma_y ) (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pure Cu</td>
<td>60</td>
</tr>
<tr>
<td>Cu = 10% Ni</td>
<td>115</td>
</tr>
<tr>
<td>Cu = 30% Ni</td>
<td>145</td>
</tr>
<tr>
<td>Cu = 70% Ni</td>
<td>170</td>
</tr>
<tr>
<td>Pure Ni</td>
<td>80</td>
</tr>
</tbody>
</table>

NOT a rule of mixtures
Yield stress data for precipitation hardened alloys

Pure Al: 25 MPa  High strength aerospace Al alloy: 500 MPa
Pure Fe: 110 MPa  Quenched/tempered high alloy (tool) steel: 2000 MPa

What particle spacing (and size) gives useful precipitation hardening?
Example: what particle spacing in Al alloy gives a yield stress increment \( \Delta \sigma_y \) of 400 MPa?

Recall for precipitation hardening:
- Increment in shear stress to bow dislocations: \( \Delta \tau_y \approx \frac{G b}{L} \)
- Yield stress increment is: \( \Delta \sigma_y \approx 3 \Delta \tau_y \)

For aluminium: shear modulus \( G = 26 \) GPa, Burgers vector \( b = 0.286 \) nm.

Hence:
\[
L \approx \frac{3 G b}{(\Delta \sigma_y)_{ppt}} \approx 55 \text{ nm}
\]
(Close to previous estimate for a volume fraction 5% of spherical particles of radius 25nm).

Consequences: processing for precipitation hardening
A few % of particles around 25nm radius gives a useful strength increment (e.g. 400MPa in Al). It is practically very difficult to manufacture solid particles this small, and to mix them into a melt before casting.

The main practical manufacturing route is to use heat treatment in the solid state, forming fine precipitates (from a solid solution) – hence the name ‘precipitation hardening’:
- Controlled temperature-time histories offer a versatile route to controlling precipitate structure, size and volume fraction (IB Materials)
- Practical precipitates vary in size from clusters of 10 or so atoms, to compounds containing 10^6 or more atoms (i.e. diameters 0.5 – 200 nm)

6.2.6 Overview: alloy processing for strength
- Many processes for making metal components (often multi-stage)
- Alloy composition & process route determine microstructure, and thus \( \sigma_y \)

Examples of alloys, applications and hardening mechanisms

<table>
<thead>
<tr>
<th>Alloy</th>
<th>Typical uses</th>
<th>Work hardening</th>
<th>Solid solution hardening</th>
<th>Precipitation hardening</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pure Al</td>
<td>Foil</td>
<td>XXX</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pure Cu</td>
<td>Wire</td>
<td>XXX</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cast Al, Mg</td>
<td>Automotive parts</td>
<td>XXX, X</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bronze (Cu-Zn)</td>
<td>X</td>
<td>XXX</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Non-heat-treatable wrought Al</td>
<td></td>
<td>XXX, XXX</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Heat-treatable wrought Al</td>
<td></td>
<td>X, X</td>
<td>XXX</td>
<td></td>
</tr>
<tr>
<td>Low carbon steels</td>
<td>Car bodies, structures, cans</td>
<td>XXX</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low alloy steels</td>
<td>Automotive parts, tools</td>
<td>X, X</td>
<td>XXX</td>
<td></td>
</tr>
<tr>
<td>Stainless steels</td>
<td>Cutlery, pressure vessels</td>
<td>XXX, XXX</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cast Ni alloys</td>
<td>Jet engine turbines</td>
<td>XXX, XXX</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

XXX: routinely used; X: sometimes used

6.3 Failure of Polymers
Polymer strength is determined by:
- Molecular architecture and bonding
- The ability of the chain molecules to unravel and slide (no real equivalent to the dislocation)
- Temperature, relative to the glass transition, and the strain-rate

Selected \( \sigma-\epsilon \) curves for polymers, at room temperature (from Materials Databook).

Amorphous Thermoplastics
(1) \( T < 0.8 T_g \): elastic-brittle
- Chain sliding limited
- Brittle fracture from inherent flaws in the material
- Little or no ductility
- e.g. PMMA (tension)
• chain mobility increases around $T_g$ as van der Waals bonds melt
• yielding takes place by crazing, shear yielding or cold drawing.

**Crazing:**
Microcracks open in tension, bridged by stiff fibres of material with aligned molecules, preventing immediate fracture.

**Cold drawing**
Polymers which do not craze can often be cold drawn. Necking occurs, but the neck is stable: the molecules align in the neck and strengthen it, so the neck spreads along the specimen.

**Elastomers:**
• elastic-brittle below $T_g$ but very large elastic strains to failure above $T_g$
• fail catastrophically with little or no ductility: elastic strain is recovered
• it is confusing (but strictly true) that the elongation to failure is zero

**Thermosets:**
• also follow the pattern of Young’s modulus
• little effect of $T_g$: slow fall in strength until the material decomposes
• above $T_g$ limited shear yielding may occur (desirable in epoxy resins used as the matrix in fibre composites)

**Summary:** Length scales of materials and microstructures
The figure summarises the microstructural features that underpin the structural (and some functional) properties of materials in the IA course.
7. Strength-limited Component Design

Selection of materials was introduced for stiffness-limited design, at minimum weight or cost. Many structural components are also strength-limited: this can be analysed following the same methodology:

1. Identify objective (e.g. minimum mass or cost)
2. Identify functional constraint (i.e. must not fail: \( \sigma_{\text{max}} < \sigma_f \))
3. Examine geometrical constraints (fixed dimensions, free variables)

### 7.1 Selection of light, strong materials

**Example:** Light, strong tensile tie

A tensile tie of specified length \( L \) is required to carry a load \( F \), without failure.

The tie has a uniform prismatic cross-section, but its area \( A \) may be varied.

**Step 1:** Objective: minimum mass

The mass is minimised by maximising the performance index:

\[
\text{mass } m = \frac{L F}{\rho} \text{ or } m \propto \frac{F}{\rho}
\]

**Step 2:** Functional constraint: must not fail, \( \sigma_{\text{max}} < \sigma_f \)

Hence strength constraint becomes:

\[
F = A \sigma_f = \text{constant}
\]

Eliminate the free variable \( A \) in the objective equation:

\[
\sigma_f = \frac{F}{A} \implies \text{mass } m = \frac{(L F)}{\rho A} \implies \text{mass } m \propto \frac{F}{\rho}\]

This is the specific strength. As with \( E/\rho \), it is commonly used to compare materials, but is not always the optimum combination.

For minimum material cost, the performance index is modified as before:

\[
\text{Cost } = \rho L A C_m \implies \text{maximise } (\sigma_f/\rho C_m)
\]

### Light, strong components in bending

Shaping the cross-section improves stiffness in bending, and also reduces the maximum stress generated by a given bending moment (IA Structures).

The effect of section shape on material selection is considered later.

To investigate the effect of strength-limited design for bending, as opposed to tension, consider material selection for a light, strong panel.

**Example:** Light, strong panel in bending

- specified span \( L \), width \( B \)
- carry load \( W \) in 3-point bending, without failure
- rectangular cross-section, depth \( D \) may be varied

Following the same procedure as before:

**Objective:** minimum mass

**Functional constraint:** Set max. stress = failure stress:

\[
\sigma_{\text{max}} = \frac{M}{I} = \frac{M}{B D^3/12}
\]

**Geometric constraint:** length \( L \), width \( B \) fixed; free variable \( D \)

**Material selection for minimum mass**

(1) On Strength – Density property chart:

Take logs as before, and re-arrange into form \( y = m x + c \):

\[
(\sigma_f/\rho) = \text{constant: lines of slope 1}
\]

\[
(\sigma_f/\rho)^{1/2} = \text{constant: lines of slope 2}
\]

(2) Apply secondary constraints (as before):

- Avoid brittle materials (ceramics, glass)
- Upper limit on cost/kg
- Environmental resistance requirements
- Manufacturing limits
- Size limits – e.g. in tension: \( A \sigma_f = \text{constant} \):

\[
\text{upper limit on } A \Rightarrow \text{lower limit on } \sigma_f
\]
7.2 Case studies in strength-limited design

(1) Materials for springs

Maximum elastic stored energy per unit volume was shown earlier: \( \sigma_f^2 / 2E \)

Property chart in Materials Databook, or in CES (Examples Paper 4).

Take care to:
(a) apply correct slope for index
(b) move line in correct direction to optimise the selection.

7.3 Effect of shape on material selection for lightweight design

Section shape is used to improve the efficiency of components and structures loaded in bending, e.g. I-beams:

(The same applies in torsion – twisting – e.g. hollow tubes).

To include shape in material selection, we need to:
- quantify the efficiency of section shape
- consider both stiffness and strength

Shape efficiency in bending: stiffness

Bending stiffness is governed by the flexural rigidity, \( EI \) (cf. IA Structures):

Stiffness, \( S = \frac{W}{\delta} = \frac{C_1 EI}{L^3} \)

where \( L = \) second moment of area

\( I = \int y^2 \, dA \)

and \( C_1 \) depends on the loading geometry.

Note that:
- stiffness, \( S \propto \) second moment of area, \( I \)
- mass (per unit length), \( mL \propto \) area, \( A \)

Shaping a section may be considered to improve efficiency in two ways:
(i) increased stiffness (\( I \)), at constant mass (\( A \))
(ii) reduced mass (\( A \)), at constant stiffness (\( I \))
Consider case (i): constant area (and mass/length):

Define shape factor, for stiffness in bending, \( \Phi_e = \frac{I}{A} \) for shaped section 
\( \frac{I}{A} \) for reference shape

A simple reference shape is a solid square section:

Hence shape factor for elastic bending stiffness:

\[ \Phi_e = \frac{12}{A^2} \]

N.B. There are physical limits to the magnitude of the shape factor: this leads to a maximum shape factor for each material (see below).

Case (ii) is more relevant to material selection: minimum mass for a given stiffness.

Recall how to derive a performance index for minimum mass, in bending:

Objective: minimum mass, \( m = \rho L A \)

Functional constraint: bending stiffness, \( S = \frac{W}{\delta} = \frac{C_1 EI}{L^3} \)

Geometric constraints: \( L \) fixed; shape and area now free variables

The stiffness constraint is: \( \frac{(W/\delta)L^3}{EI} = \text{constant} \)

Substituting for \( I \), using the shape factor, \( \Phi_e \):

\[ \frac{(W/\delta)L^3}{EI} = \frac{E\Phi_e A^2}{C_1} \]

Hence area \( A \sim \frac{1}{(E\Phi_e)^{1/2}} \)

Substituting into objective equation: mass, \( m \sim \frac{\rho}{(E\Phi_e)^{1/2}} \)

Hence for minimum mass, maximise performance index: \( \left( \frac{E\Phi_e}{\rho} \right)^{1/2} \)

For the same stiffness, shaping a section reduces the mass (relative to a solid square section) by a factor of \( 1/(E\Phi_e)^{1/2} \).

Maximum shape factor: stiffness

The maximum shape factor depends on the physical limits on section thickness due to:

- the capabilities of manufacturing processes
- buckling failure of thin-walled sections

Key point: different materials can be shaped to a different extent.

<table>
<thead>
<tr>
<th>Material</th>
<th>Typical maximum shape factor, ( \Phi_e )</th>
<th>Typical mass ratio by shaping, ( \frac{1}{(E\Phi_e)^{1/2}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steels</td>
<td>64</td>
<td>1/8</td>
</tr>
<tr>
<td>Al alloys</td>
<td>49</td>
<td>1/7</td>
</tr>
<tr>
<td>Fibre Composites</td>
<td>36</td>
<td>1/6</td>
</tr>
<tr>
<td>Wood</td>
<td>9</td>
<td>1/3</td>
</tr>
</tbody>
</table>

Numerical values for performance index, with and without shape:

For constant \( \Phi_e \), the shape is fixed:

\[ \frac{E^{1/2}}{\rho} \]

i.e. as area varies, the dimensions remain in constant proportion

In this case the performance index becomes:

<table>
<thead>
<tr>
<th>Material</th>
<th>Index with fixed shape, ( E^{1/2}/\rho )</th>
<th>Index including max. shape factor, ( E^{1/2}/\rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steels</td>
<td>1.88</td>
<td>14.9</td>
</tr>
<tr>
<td>Al alloys</td>
<td>3.10</td>
<td>21.7</td>
</tr>
<tr>
<td>CFRP</td>
<td>6.35</td>
<td>37.5</td>
</tr>
<tr>
<td>Wood</td>
<td>4.84</td>
<td>14.5</td>
</tr>
</tbody>
</table>

Notes:

- Composites lose some of their performance advantage over metals
- Wood falls behind in applications which can exploit shape

Shape factor for bending strength

Similar arguments apply to quantify the effect of shape on strength.

Bending strength is governed by the maximum moment, \( M \), and the corresponding maximum stress, \( \sigma \) (cf. IA Structures):

\[ \sigma_{\text{max}} = \frac{M}{I} \] where \( I \) = second moment of area

Failure moment:

\[ M_f = \frac{l}{\gamma_{\text{max}}} \sigma_f = Z_e \sigma_f \]

where \( Z_e \) is the elastic section modulus.
Define shape factor, for strength in bending, \( \phi_f = \frac{Z_e}{Z_s} \) for shaped section, \( \phi_f = \frac{Z_e}{Z_s} \) for reference shape

Using the same reference shape of a solid square section: \( \phi_f = \frac{6}{A^{3/2}} \)

For minimum mass, maximise performance index: \( \frac{(\sigma_t \phi_f)^{2/3}}{\rho} \)

(derivations on Examples Paper 4).

Notes:
- for the same strength, shaping reduces the mass (relative to a solid square section) by a factor of \( 1/(\phi_f)^{3/2} \).
- for constant \( \phi_f \) (fixed shape), the index becomes \( \sigma^{2/3}/\rho \).

Maximum shape factor: strength

The same physical limits on section thickness determine the maximum shape factor for strength, for each material class:

<table>
<thead>
<tr>
<th>Material</th>
<th>Typical maximum shape factor, ( \phi_f )</th>
<th>Typical mass ratio by shaping, ( 1/(\phi_f)^{3/2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steels</td>
<td>13</td>
<td>0.18</td>
</tr>
<tr>
<td>Al alloys</td>
<td>10</td>
<td>0.22</td>
</tr>
<tr>
<td>Fibre Composites</td>
<td>9</td>
<td>0.23</td>
</tr>
<tr>
<td>Wood</td>
<td>3</td>
<td>0.48</td>
</tr>
</tbody>
</table>

Notes:
- Shaping has a smaller influence on strength than on stiffness (because increasing \( I \) is partly achieved by increasing \( Y_{max} \))
- Metals again catch up a little with composites; wood falls behind.

Summary: solving problems with shape

The differences in shape factor between materials are of comparable magnitude to the differences in modulus and strength:
- shape is significant in material selection for bending applications.

Hence if area and shape can both be varied:
- either, use performance indices including shape factor
- or, use performance indices without shape, but comment on likely effect of shape (metals > composites > wood)

Case study: Plastic for lightweight bicycles?

Bike frames are limited by both stiffness and strength, and may be optimised for low mass or low cost, depending on the market. Full analysis of problems with more than one functional constraint is discussed below. A preliminary analysis may be conducted to consider the question: would a plastic bicycle be lightweight?

Assume the following:
- frame loading is dominated by bending
- shape is fixed (e.g. tubes of given radius/thickness ratio), size may vary
- recall the relevant performance indices to maximise for minimum mass are:
  - \( E^{1/2}/\rho \) for given stiffness,
  - \( \sigma^{2/3}/\rho \) to avoid failure

Plot these indices against one another in CES.

Notes:
- strong competition between Al, Mg, Ti alloys, Al-SiC and GFRP
- steels do not perform well for low weight: CFRP is outstanding
- polymers cannot compete, particularly on stiffness
- wood performs well, but cannot in practice be made into thin-walled tubes
## 7.4 Material selection with multiple constraints

In earlier examples of lightweight design (with fixed shape):
- objective: minimum mass
- either, stiffness-limited → functional constraint: given stiffness
- or, strength-limited → functional constraint: avoid failure

### Performance index analysis

(e.g. lightweight cantilever in bending):
- assume solid circular section, radius $R$
- fixed length $L$, specified end load $F$
- and allowable tip deflection $\delta$
- OR, must not fail

#### Stiffness-limited

$$\sigma_{\text{max}} = \frac{3F L}{2\pi R^2}$$

#### Strength-limited

$$\sigma_f = \frac{FL}{\pi R^4}$$

Eliminating $R$:
- $$\sigma_{\text{max}} = \frac{3FL}{2\pi \rho}$$
- $$\sigma_f = \frac{FL}{\pi \rho}$$

### If one constraint (stiffness or strength):
- minimise mass → maximise appropriate index ($E^{1/2}/\rho$ or $\sigma_f^{2/3}/\rho$)
- do not need values for $F$, $L$, $\delta$

### If both constraints apply:
- limiting mass is the higher of $m_{\delta}$ and $m_{\sigma}$ (to guarantee both are met)
- must evaluate actual masses ⇒ need values for $F$, $L$, $\delta$
- lightest material: the lowest of the limiting mass values

### Example: cantilever with stiffness and strength constraint

Fixed length $L = 0.5m$, end load $F = 500N$
Allowable deflection $\delta = 50mm$; must not fail (max. stress $< \sigma_f$)

<table>
<thead>
<tr>
<th>Material</th>
<th>$E$ (GPa)</th>
<th>$\rho$ (kg/m$^3$)</th>
<th>$\sigma_f$ (MPa)</th>
<th>$m_{\delta}$ (kg)</th>
<th>$m_{\sigma}$ (kg)</th>
<th>Design-limiting constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>CFRP</td>
<td>120</td>
<td>1500</td>
<td>600</td>
<td>0.16</td>
<td>0.15</td>
<td>Stiffness</td>
</tr>
<tr>
<td>Ti alloy</td>
<td>120</td>
<td>4000</td>
<td>700</td>
<td>0.47</td>
<td>0.42</td>
<td>Stiffness</td>
</tr>
<tr>
<td>Al alloy</td>
<td>70</td>
<td>2700</td>
<td>600</td>
<td>0.37</td>
<td>0.36</td>
<td>Stiffness</td>
</tr>
<tr>
<td>Alloy steel</td>
<td>210</td>
<td>7800</td>
<td>600</td>
<td>0.62</td>
<td>0.60</td>
<td>Strength</td>
</tr>
<tr>
<td>Nylon</td>
<td>3</td>
<td>1100</td>
<td>100</td>
<td>0.73</td>
<td>0.37</td>
<td>Stiffness</td>
</tr>
<tr>
<td>Wood</td>
<td>12</td>
<td>600</td>
<td>70</td>
<td>0.20</td>
<td>0.26</td>
<td>Strength</td>
</tr>
</tbody>
</table>

**Result**: CFRP is the lightest material

Notes:
- in this example, CFRP was the lightest material for both constraints
- this is not always the case: the best material may not in fact be the lightest on either criterion

### Further refinements

**Minimum cost**
- same analysis, with each limiting mass × cost/kg to convert to cost
- same design-limiting criterion will apply for each material

<table>
<thead>
<tr>
<th>Material</th>
<th>$m$ (kg)</th>
<th>$C_m$ (£/kg)</th>
<th>Cost (£)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CFRP</td>
<td>0.16</td>
<td>60</td>
<td>9.6</td>
</tr>
<tr>
<td>Ti alloy</td>
<td>0.47</td>
<td>40</td>
<td>18.8</td>
</tr>
<tr>
<td>Al alloy</td>
<td>0.37</td>
<td>2</td>
<td>0.74</td>
</tr>
<tr>
<td>Alloy steel</td>
<td>0.80</td>
<td>1</td>
<td>0.80</td>
</tr>
<tr>
<td>Nylon</td>
<td>0.73</td>
<td>4</td>
<td>2.92</td>
</tr>
<tr>
<td>Wood</td>
<td>0.26</td>
<td>2</td>
<td>0.52</td>
</tr>
</tbody>
</table>

**Result**: Wood is the cheapest material

**Secondary constraints**
- as in earlier examples: comment on toughness, corrosion, manufacturing, joining etc.